DiffServ Pricing Games in Multi-class Queueing Network Models

Parijat Dube & Rahul Jain

***Abstract*— Introduction of differentiated services on the Internet has failed primarily due to many economic impediments. We focus on the provider competition aspect, and develop a multi-class queueing network game framework to study it. Each network service provider is modeled as a single-server multi-class queue. Providers post prices for various service classes. Traffic is elastic and there are multiple types of it, each traffic-type is sensitive to a different degree to Quality of Service (QoS). Arriving users choose a provider and a class for service. We study the pricing and service competition between the providers in a game-theoretic setting. We provide sufficient conditions for the existence of Nash equilibrium in the Bertrand (pricing) game between the multi-class queueing service providers. We also characterize the inefficiency (price of anarchy) due to strategic DiffServ pricing.**

***Index Terms*— DiffServ, Queueing networks, Bertrand game, Nash equilibrium, Price of Anarchy.**

I. Introduction

It is well-known that introduction of service differentiation in networks has failed not due to inadequacies of technological solutions (e.g., DiffServ) but primarily due to economic impediments. At the same time, this became part of the network neutrality debate wherein network service providers (NSPs) (e.g., AT&T, Verizon, etc.) want to introduce service differentiation through price differentiation while content providers (such as Google, Yahoo, etc.) are opposed to such an arrangement. Both sides have made various arguments for and against network neutrality and service differentiation whose veracity has been hard to judge. Motivated by this, we propose a simple queueing model which can be useful in studying some of these questions. We caution that the larger policy issue is a lot more complicated and we cannot hope to have all the answers by studying simple models. Nevertheless, the framework can be used to address some basic questions: Is service differentiation better for 'network utility maximization'? How much is lost in network utilization due to competition between providers when they offer differentiated services?

Parijat Dube is with the IBM T.J. Watson Research, Hawthorne, NY 10532. Email: pdube@us.ibm.com.

Rahul Jain is with the University of Southern California, Los Angeles, CA 90089. Email: rahul.jain@usc.edu. His research is supported by a James H. Zumberge Faculty Research and Innovation award, the NSF NetSE grant IIS-0917410 and the NSF CAREER award CNS-0954116.

We ignore the internal topology of a network and think of each autonomous network of an NSP as a node represented by its "Norton-equivalent" queue. There are N providers each of whom operates a GI/G/1-PS queue with two classes with processor-sharing between them (we restrict to queues with convex delay functions). There are two types of traffic, type 1 traffic is more delay-sensitive than type 2. Users of each type arrive according to a renewal process (e.g., Poisson). Providers could charge a higher service price for class 1 than for class 2. Upon arrival, each user then has to decide which provider/queue to join (type i users only go to class i). If it joins a particular queue, it has to pay a service price to the operator. Each user is also delay sensitive and his service utility is a non-increasing function of the delay it suffers. Thus, he joins a queue that offers the maximum net utility, i.e., service utility minus delay cost and price. We will assume that the queues are *partially observable* in the sense that expected delay metrics are available to the user upon his arrival but not the instantaneous queue lengths. Each provider's objective on the other hand is to maximize his total payoff. The players are competitive and strategic. The objective of each provider is to pick service prices for the two classes as well as the *processor-sharing ratio* that maximizes his total payoff taking into account the competition provided by the other providers. The question at hand is what kind of outcomes may one expect from the competition between the providers? And are the outcomes optimal or efficient in some sense?

There has been a plethora of work on network pricing, most of it concerned with using pricing models for design and analysis of network flow control. There is a huge literature on economic models of networks as well (see [3]) including pricing of Internet services [16], and in particular, DiffServ services as well [26], [17], [18], [1], [2], [25], [24]. Our focus, however, is different. We ask what impact competitive and strategic DiffServ pricing has on network utility maximization? Since QoS is an important metric of network utility, we explicitly introduce a queueing network model. We believe such a multi-class queueing network game model is new in the study of DiffServ. We provide a brief survey of related work.

Pricing as a means to queueing stability was first studied in [22] with Poisson arrivals to an exponential server queue [6]. In [7], a scheduling policy was derived for a multi-class queue that maximizes the net utility. Mendelson and Whang [20] introduced a stylized model for a single queuing service provider that has been widely used. They introduced an incentive-compatible priority pricing rule for the M/M/1 queue which maximizes the social welfare. In [10], a *cost sharing* perspective is taken on sharing the total delay cost, and the Aumann-Shapley mechanism is used to determine individual user payments. While single server models have been much analyzed [8], multiple queueing service provider models are not so well understood. The earliest work on this is [15] while a variation of the Mendelson-Whang model for two identical servers (with unobservable queues) was considered in [14]. Both of these established the suboptimality of equilibrium flows under various settings - a Bertrand (pricing) game and a Cournot (capacity) game.

More recently, [9] considered the Bertrand game between multiple NSPs with linear affine delay functions (a very rough approximation for delay in queueing networks) and showed the game to have a price of anarchy (PoA) of 1/3.125. This result was sharpened by [23], [21] to 2/3 for general concave delay functions. While a bound on PoA was obtained, it was conjectured that a Nash equilibrium may not exist in general. In [5], [12], sufficient conditions for existence and uniqueness of Nash equilibrium are given. All these works focused on single class queues only. Our recent work [11] does study multi-class queues but there is only one type of traffic. Earlier in [5], we have addressed existence of equilibrium in single-class queueing games where providers can choose both prices as well as service capacities. In [4], the authors considered the case of inelastic traffic each arriving user must choose among two M/G/1 queues (cannot balk). There, a Stackelberg game was considered and it was shown that an equilibrium does not exist, but an oscillatory "price wars" behavior around a limit point was noticed.

The focus in this paper is on understanding DiffServ pricing in a multi-provider environment. We introduce a multi-class queueing network with elastic traffic of multiple types. Providers compete on setting prices for various classes. We give sufficient conditions for existence of Nash equilibrium and show that the price of anarchy with DiffServ classes in networks is the same as in the absence of DiffServ [21], [23]. This simple model seems to suggest that no (worst-case) welfare gains may be made by introduction of service differentiation.

II. Model and Preliminaries

We consider N providers each of whom offers a queued service to customers. Each provider operates a queue with two classes and processor sharing between them, and convex delay functions for each class. There are two types of traffic with traffic with different sensitivity to delay. Traffic of type 1 always goes into class 1 of the queues, traffic of type 2 always goes into class 2 of the queues.

As illustrated in Figure 1, let x_{ki} be the traffic flow of type k into the ith queue in the kth class. Provider i has service capacity y_i and he splits his service capacity as $\alpha_i y_i$ for class 1 and $\bar{\alpha}_i y_i$ for class 2 by doing processor-sharing where $\bar{\alpha}_i = 1 - \alpha_i$. Service quality metrics are delay functions $d_{ik}(x_{ki}; \alpha_i)$ for class k where delays at $\alpha_i = 0$, $d_{ik}(x_{ki}; 0)$ will be assumed to be infinite. A higher value of d_{ik} implies a degradation in service quality. If the delay suffered by a unit flow of type k is d, then the perceived delay cost per unit is $\theta_k d$. Providers charge prices $p_i = (p_{i1}, p_{i2})$ per unit flow.

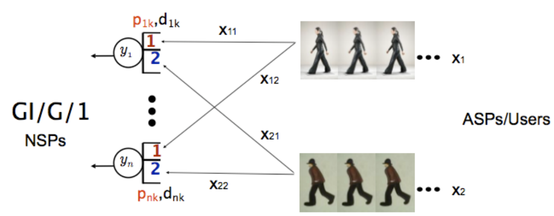

Fig. 1. DiffServ pricing game between multi-class queued service providers

We assume that each arriving user is infinitesimal and non-strategic (i.e., price-taker) and their preferences can be aggregated so that the *aggregate utility* derived by all users of type k is $V_k(\sum_i x_{ki})$. Note for prices p_{ik} and delays d_{ik}, the total demand or flow of type k can be given by $z_k = (V_k')^{-1}(\min_i\{p_{ik} + \theta_k d_{ik}\})$. The $p_{ik} + \theta_k d_{ik}$ term shall be called the *full price* of queue i and class k. We will take the net *social welfare* to be

$$S(x, \alpha) := \sum_k V_k(\sum_i x_{ki}) - \sum_{k,i} \theta_k x_{ki} d_{ik}(x_{ki}; \alpha_i).$$

We will call a traffic vector x^{**} to be *socially efficient* if it maximizes the social welfare maximization problem $\max_{x,\alpha} S(x, \alpha)$, where $\alpha_i \in [0, 1]$ and $x_{ki} \geq 0, \ \forall i, k$. We will make the following assumptions.

Assumptions. (A1) $V_k(x_k)$ is strictly increasing and concave in $x_k = \sum_i x_{ki}$.

(A2) $d_{ik}(x_{ki}; \alpha_i)$ are strictly increasing and convex in x_{ki} for given α_i.

(A3) $V_k \in C^3$, and $d_{ik} \in C^2$, i.e., V_k'' and d_{ik}' are continuous.

(A4) $V_k'(\sum_i \alpha_{ik} y_i) \geq \theta_k d_{ik}(0, \alpha_{ik}),\ \forall i, k$, with $\alpha_{i,k} > 0$. Note that for two players $\alpha_i = \alpha_{i1}$ and $\bar{\alpha}_i = \alpha_{i2}$.

These assumptions are justified in [5]. We first characterize the rate vector to be socially optimal.

Lemma 1: Under Assumptions (A1)-(A2), a socially efficient rate vector x^{**} exists and is characterized by: $\forall i, k$,

$$V_k'(\sum_j x_{kj}^{**}) - x_{ki}^{**}\theta_k d_{ik}'(x_{ki}^{**}) - \theta_k d_{ik}(x_{ki}^{**}) \leq 0. \quad (1)$$

with equality when $x_{ki}^{**} > 0$.

A user of type k sends the marginal traffic (e.g. a packet in the communication network) to queue Q_i in class k if $i \in \arg\min_j p_{jk} + \theta_k d_{jk}(x_{kj}; \alpha_j)$. We assume that each customer type sends traffic at some rate x being determined by the prices and the delay cost it faces. Thus, a natural outcome of such decision-making by each user is the following.

Definition 1 (Wardrop equilibrium): For a price vector p, and processor-sharing ratio vector α, a flow vector $x^*(p, \alpha)$ is a *Wardrop equilibrium* if and only if it satisfies the following $\forall i, k$,

$$V_k'(\sum_i x_{ki}^*) - p_{ik} - \theta_k d_{ik}(x_{ki}^*; \alpha_i) \leq 0. \quad (2)$$

with equality when $x_{ki}^* > 0$.

We will assume that instantaneous queue lengths are not observable but the expected delays in each class for each queue is available to all arriving users. We will assume that the queues have infinite buffers for every class. For provider i, if total traffic is x_i with prices p, then his *net payoff* is

$$\Pi_i(p_i, \alpha_i; p_{-i}, \alpha_{-i}) = \sum_k p_{ik} x_{ki}(p_{ik}, \alpha_i, p_{-i,k}, \alpha_{-i}).$$

Thus, given prices of other providers, he must pick a price vector p_i that maximizes his total payoff.

A natural question now is does there exist an equilibrium where providers announce expected delay guarantees, and users then choose traffic rates and the system equilibrates through some prices and traffic rates (such that the full price is the same for all queues in all classes), delays guarantees offered are met, and all providers are satisfied at these prices? When players are non-strategic, then the existence of *competitive equilibrium* was established in [27]. When players are strategic however, we must look at Nash equilibrium.

Definition 2 (Nash equilibrium): A price vector and processor-sharing ratio (p^*, α^*) is a Nash equilibrium if (i) the corresponding Wardrop equilibrium is $x^*(p^*, \alpha^*)$, and (ii) for each i, given $(p_{-i}^*, \alpha_{-i}^*)$,

$$(p_i^*, \alpha_i^*) \in \arg\max_{p_i, \alpha_i} \sum_k p_{ik} x_{ki}^*(p_i, \alpha_i, p_{-i}^*, \alpha_{-i}^*). \quad (3)$$

III. DiffServ Pricing Game

When the providers are strategic, they try to anticipate the actions of the other players (providers) and it may not be possible to achieve socially optimal allocations. The providers strategize by picking prices. Capacity available to them will be assumed fixed. We study equilibria in such a pricing game (also called a Bertrand game) between queueing service providers. In previous work, we considered single-class queues, and established existence of Nash equilibrium [5]. The price of anarchy in that setting had already been established in [23], [21]. We also considered the setting where providers choose capacity and prices at the same time [5].

Here we consider a model more appropriate to study DiffServ. There are multiple traffic types. Each queue also has multiple classes with processor-sharing between the classes. For simplicity, we present results for only two traffic classes (delay-sensitive and best-effort) and two corresponding classes in each queue but these can be extended to a general setting at the cost of more complicated analysis.

The allocations for a given price vector p and processor-sharing ratio α is a Wardrop equilibrium, and from (2) we have $\forall i$,

$$V_k'(\sum_j x_{kj}) - \theta_k d_{ik}(x_{ki}; \alpha_i) - p_{ik} \leq 0, \quad (4)$$

with equality when $x_{ki} > 0$. Furthermore, the prices p_i and processor-sharing ratio α_i which maximize the aggregate payoff of provider i, $\Pi_i = \sum_k p_{ik} x_{ki}$, must satisfy,

$$x_{ki} + p_{ik}\frac{\partial x_{ki}}{\partial p_{ik}} = 0, \quad p_{i1}\frac{\partial x_{1i}}{\partial \alpha_i} + p_{i2}\frac{\partial x_{2i}}{\partial \alpha_i} = 0. \quad (5)$$

Note that we also must establish the joint concavity of Π_i in (p_i, α_i) in terms of conditions on second order derivatives.

Let (x^*, p^*, α^*) satisfy (4) and (5). Then, (p^*, α^*) is a Nash equilibrium. We denote by $U_i^* = \sum_k p_{ik}^* x_{ki}^*$, the payoff of provider i at this Nash equilibrium.

Lemma 2: Under Assumptions (A1)-(A4), if (p^*, α^*) is a Nash equilibrium, then there exists a Nash equilibrium $(\tilde{p}, \alpha^*)$ with equality in (4).

Proof: Let $x^* = x^*(p^*, \alpha^*)$ be a WE. If $x_{ki}^* > 0$, define $\tilde{p}_{ik} = p_{ik}^*$. If $x_{ki}^* = 0$ and there is an inequality in (4), define

$$\tilde{p}_{ik} = V_k'(\sum_j x_{kj}^*) - \theta_k d_{ik}(0; \alpha_i^*).$$

Note this is always non-negative by Assumption (A4). With price $\tilde{p}$, the allocation is still x^*, implying that the corresponding payoff $\tilde{U}_i = \sum_k \tilde{p}_{ik} x_{ki}^*$ is optimal and equal to U_i^* for all i. Further $(x^*, \tilde{p})$ also solves (4) with equality. Thus, $(\tilde{p}, \alpha^*)$ is a Nash equilibrium. ■

Remark 1. Note that we did not use convexity of delay functions assumed in Assumption (A2).

Remark 2. In subsequent discussion, without loss of generality, we absorb θ_k into the delay function, d_{ik}.

The implication of this lemma is that if a Nash equilibrium exists, then we can look for a Nash equilibrium of type $\tilde{p}$ for which we have equality for every i in (4). We can establish existence of such a Nash equilibrium under certain conditions.

Theorem 1: If Π_i is jointly concave in $(p_{i1}, p_{i2}, \alpha_i)$ for every i then under Assumptions (A1)-(A4), there exists a Nash equilibrium in the DiffServ Bertrand game. The proof is relegated to the appendix.

A. Concavity of the payoff function

We next establish the joint concavity of revenue of provider i in $(p_{i1}, p_{i2}, \alpha_i)$. To show this we need to establish negative definiteness/semi-definiteness of the Hessian for provider i revenue using the following known result.

Lemma 3: A $n \times n$ symmetric matrix is
(i) negative definite iff the k *leading priniciple minors* are < 0 if k is odd and > 0 if k is even.
(ii) negative semidefinite iff *all the kth order principle minors* of are ≤ 0 if k is odd and ≥ 0 if k is even.

The Hessian for provider's optimization is given by

$$\mathscr{H}_{\Pi_i} = \begin{bmatrix} \Pi_{i,p_{i1}^2} & \Pi_{i,p_{i1},p_{i2}} & \Pi_{i,p_{i1},\alpha_i} \\ \Pi_{i,p_{i1},p_{i2}} & \Pi_{i,p_{i2}^2} & \Pi_{i,p_{i2},\alpha_i} \\ \Pi_{i,p_{i1},\alpha_i} & \Pi_{i,p_{i2},\alpha_i} & \Pi_{i,\alpha_i^2} \end{bmatrix} \tag{6}$$

By repeated partial differentiation of Π_i, we get: $\Pi_{i,p_{ik}^2} = 2x_{ki,p_{ik}} + p_{ik}x_{ki,p_{ki}^2}$; $\Pi_{i,p_{ik},p_{ij}} = 0, j \neq k$; $\Pi_{i,\alpha_i^2} = p_{i1}x_{1i,\alpha_i^2} + p_{i2}x_{2i,\bar{\alpha}^2}$; $\Pi_{i,p_{i1},\alpha_i} = x_{1i,\alpha_i} + p_{i1}x_{1i,p_{i1},\alpha_i}$; $\Pi_{i,p_{i2},\alpha_i} = -x_{2i,\bar{\alpha}_i} - p_{i2}x_{2i,p_{i2},\bar{\alpha}_i}$.

Define $\mathcal{H}_1 = \Pi_{i,p_{i2}^2}\Pi_{i,\alpha_i^2} - \Pi^2_{i,p_{i2},\alpha_i}$, $\mathcal{H}_2 = \Pi_{i,p_{i1}^2}\Pi_{i,\alpha_i^2} - \Pi^2_{i,p_{i1},\alpha_i}$, $\mathcal{H}_3 = \Pi_{i,p_{i1}^2}\Pi_{i,p_{i2}^2}$. Thus we have

Proposition 1: If Π_i is (marginally) concave in $p_{ik}, k = 1,2$ and α_i, and derivatives of the profit function satisfy:

$$\Pi_{i,p_{ik}^2}\Pi_{i,\alpha_i^2} \geq \Pi^2_{i,p_{ik},\alpha_i} \quad k = 1,2 \tag{7}$$

$$\Pi_{i,p_{i1}^2}\mathcal{H}_1 - \Pi^2_{i,p_{i1},\alpha_i}\Pi_{i,p_{i2}^2} \leq 0 \tag{8}$$

then Π_i is *jointly concave* in $(p_{i1}, p_{i2}, \alpha_i)$.

Proof: Concavity of Π_i in p_{ik} and α_i ensures that the first order leading principal minors of $\mathscr{H}_{\Pi_i}$ are negative. The three second order principal minors of $\mathscr{H}_{\Pi_i}$ are positive, $\mathcal{H}_1$ and $\mathcal{H}_2$ positive by (7) and $\mathcal{H}_3$ by concavity of Π_i in p_{ik}. (8) makes the third order principal diagonal of $\mathscr{H}_{\Pi_i}$, i.e., the determinant of $\mathcal{H}$, negative. Thus from Lemma 3, we have the joint concavity of Π_i. ■

Furthermore, if Π_i is jointly concave in $(p_{i1}, p_{i2}, \alpha_i)$, then the provider optimization problem is convex, and an optimum always exists.

B. Special Case: Linear Utility

We consider the case with linear utilities. We establish sufficient conditions for the concavity of Π_i in p_{ik} and α_i for this case. Let $\alpha_{i1} = \alpha_i$ and $\alpha_{i2} = 1 - \alpha_i$.

Lemma 4: Under Assumptions (A1)-(A4), if d_{ik} is strictly decreasing in α_{ik} then x_{ki} is strictly decreasing in p_{ik} and strictly increasing in α_{ik} for $k = 1(2)$.

Proof: Without loss of generality, we can take $i = 1$. From (26) in Appendix we have $x_{k1,p_{1k}} = -d^{-1}_{1k,x_{k1}}\frac{\Delta_{k,-1}}{\Delta_k}$. From Assumption (A1) and (A2) we have $\Delta_{k,-1}, \Delta_k > 0$ and $d_{1k,x_{k1}} > 0$. Thus $x_{k1,p_{1k}} < 0$. From (27) we have $x_{11,\alpha_1} = -d^{-1}_{11,x_{11}}d^{-1}_{11,\alpha_1}\frac{\Delta_{1,-1}}{\Delta_1}$. Since $d_{11,x_{11}} > 0$, $d_{12,\alpha_1} < 0$, we have $x_{11,\alpha_1} > 0$. Similarly we can establish $x_{21,\alpha_1} < 0$. ■

We next use Lemma 4 to establish the following.

Proposition 2: Under Assumptions (A1)-(A4), if V_k is linear and d_{ik} is increasing in α_{ik} and jointly convex in (x_{ik}, α_i) then x_{ki} and Π_i are (marginally) concave in p_{ik} and α_{ik}.

Proof: Wlog we take $i = 1$. For linear V_k, with $V_k'' = 0$, we have $\Delta_{k,-1} = \Delta_k = 1$. By taking derivative with p_{ik} in (26) we get $x_{11,p_{11}^2} = \frac{-d_{11,x_{11}^2}x^2_{11,p_{11}}}{d_{11,x_{11}}}$. Thus $x_{11,p_{11}^2} \leq 0$ as $d_{11,x_{11}^2} \geq 0$ (convexity) and $d_{11,x_{11}} > 0$. Similarly we can establish $x_{21,p_{12}}$ is concave in p_{12}. By taking derivative with α_i of (27) we get:

$$d_{11,x_{11}}x_{11,\alpha_1^2} + d_{11,x_{11},\alpha_1}x_{11,\alpha_1} + d_{11,\alpha_1^2} = 0.$$

From Prop. 4 we have $x_{11,\alpha_1} > 0$. Joint convexity of d_{11} implies $d_{11,\alpha_1^2} \geq 0$ and

$$d^2_{11,x_{11},\alpha_1} \quad \geq \quad d_{11,x_{11}^2}d_{11,\alpha_1^2} \geq 0.$$

Simlarly we can establish concavity of x_{21} in p_{12} and α_1. Then the concavity of Π_i in p_{ik} and α_i follows from expressions for Π_{i,p_{ik}^2} and Π_{i,α_i^2} obtained before Prop. 1. ■

Remark 3: Observe that M/M/1 type delay functions are jointly convex in flow and capacity and can be candidate delays for Prop. 2.

We use Prop. 2 to establish joint concavity of Π_i in the next theorem which using Prop. 1 establishes existence of Nash equilibrium.

Theorem 2: Under the assumptions of Prop. 2, if V_k' and derivatives of d_{ik} also satisfy:

$$d^{-1}_{ik,\alpha_i} = (d_{ik} - V_k')\left(\frac{d_{ik,x_{ki},\alpha_1}}{d_{ik,x_{ki}}}\right), \tag{9}$$

then Π_i is jointly concave in $(p_{i1}, p_{i2}, \alpha_i)$.

Proof: Since Π_i is concave in p_{ik} and α_i from Prop. 2, to establish joint concavity of Π_i (7) and(8) needs to be satisfied. Observe that if $\Pi_{i,p_{ik},\alpha_i} = 0$, then (7) is satisfied as Π_i is concave in p_{ik} and α_i. Also (8) is satisfied as $\mathcal{H}_1 \geq 0$. For $\Pi_{i,p_{ik},\alpha_i} = 0$ we need to show:

$$x_{ki,\alpha_i} + p_{ik} x_{ki,p_{ik},\alpha_i} = 0, \;\; \forall k = 1, 2. \tag{10}$$

Taking derivative of (26) with respect to α_i we get, with linear utilities:

$$d_{ik,x_{ki}} x_{ki,p_{ik},\alpha_i} + d_{ik,x_{ki},\alpha_i} x_{ki,p_{ik}} \;\; = \;\; 0. \tag{11}$$

Substituting x_{ki,p_{ik},α_i} from (11) in (10) we get:

$$x_{ki,\alpha_i} \;\; = \;\; -p_{ik} \left(\frac{d_{ik,x_{ki},\alpha_i}}{d_{ik,x_{ki}}} \right) x_{ki,p_ik}. \tag{12}$$

Substituting expressions for x_{ki,α_i} from (27), $x_{ki,p_{ik}}$ from (26) in (12) and with $p_{ik} = V'_k - d_{ik}$ (Wardrop equilibrium condition) we get (9). ■

IV. Price of Anarchy of the DiffServ Game

An important question now is what is lost due to competitive, strategic pricing in DiffServ networks as compared to the social-welfare maximizing optimum. A relevant metric is the *price of anarchy* (PoA) which we now define. Given a social welfare function $S(x(p))$ for strategy vector p, we define the *Price of Anarchy* as

$$\eta = \min_{p^* \text{ is N.E.}} \frac{S(x(p^*, \alpha^*))}{S(x^{**})}.$$

In this section, we compute the PoA of the DiffServ pricing game and compare it to the PoA of the single-class queueing game. We denote partial derivatives, $d_{ik,x_{ki}}$ by d'_{ik}. We first find the prices at Nash equilibrium and social optimum.

Proposition 3: (i) At Nash equilibrium, the prices are given by

$$p_{ik} = d'_{ik} x_{ki} + \delta_{ik} x_{ki}, \;\; \forall i, k = 1, 2, \tag{13}$$

where, $\delta_{ik} = (\sum_{j \neq i} (d'_{jk})^{-1} - (V'')^{-1})^{-1}$, and flows x_{ki} satisfy

$$x_{1i}/x_{2i} = d_{i2,\bar{\alpha}_i}/d_{i1,\alpha_i}.$$

(ii) The socially optimal prices are given by

$$p_{ik} = x_{ki} d'_{ik}, \;\; \forall i, k = 1, 2.$$

Proof: (i) We can write a provider's optimization problem as maximizing his payoff subject to his prices and all flows satisfying the Wardrop and Nash equilibrium conditions. Thus,

$$\begin{aligned} \max_{p_i, x \geq 0, 0 \leq \alpha_i \leq 1} \quad & p_{i1} x_{1i} + p_{i2} x_{2i} \\ \text{s.t. } p_{ik} + d_{ik} \;\; &= \;\; V'_k, \\ p_{jk} + d_{jk} \;\; &= \;\; p_{ik} + d_{ik}, \forall j, j \neq i \end{aligned} \tag{14}$$

Writing the KKT conditions for the above, we get for $k = 1, 2$:

$$x_{ki} + \lambda_{ik} - \sum_{j \neq i} \lambda_{jk} = 0, \tag{15}$$

$$p_{ik} + \lambda_{ik}(d'_{ik} - V''_k) - \sum_{j \neq i} \lambda_{jk} d'_{ik} = 0, \tag{16}$$

$$-\lambda_{ik} V''_k + \lambda_{jk} d'_{jk} = 0, \quad j \neq i, \tag{17}$$

From (15), (16) we have $p_{ik} = x_{ki} d'_{ik} + \lambda_{ik} V''_k$. Further, from (15) and (17), we get $\lambda_{i1} = \frac{x_{ki}}{\sum_{j \neq i} \frac{V''}{d'_{jk}} - 1}$, which implies the desired result.

(ii) Differentiating $S(x) = \sum_k V_k(\sum_i x_{ki}) - \sum_{i,k} d_{ik} x_{ki}$, w.r.t x_{ki} we get $V'_k - d_{ik} - x_{ki} d'_{ik} = 0$, thus $p_{ik} = x_{ki} d'_{ik}$. ■

Using the result above, we can now obtain a lower bound on the PoA of the DiffServ game. Surprisingly, it is the same as for the single-class queueing game.

Theorem 3: For the price and capacity game with affine delays, under Assumptions (A1)-(A4), if there exists a pure strategy Nash equilibrium, then the price of anarchy is at least 2/3.

The proof is in the appendix.

V. Conclusion

One of the fundamental challenges for future Internet architecture evolution is introduction of Quality-of Service (QoS). This is not because current Internet architecture cannot support QoS but because of economic impediments. In this paper, we introduce a simple multi-class queueing network model that can be used to study such issues. In such a queueing game model we have established the following:

(i) we gave sufficient conditions for existence of Nash equilibrium with processor-sharing among classes, and

(ii) we showed that the price of anarchy of the pricing game is 2/3.

This happens to be the same as in a single-class setting. This seems to suggest that as far as worst-case equilibrium social welfare is concerned, DiffServ will not result in an improvement, though there may be many other reasons for its adoption.

These results are preliminary, and based on a simple model. Thus, we advise caution on the conclusion until the same result is achieved in more varied models. In future work, we will extend this framework to more general queueing networks (than just parallel queues) which are more relevant because implementation of DiffServ requires QoS SLAs between peering network providers. We will also study the game when multi-class providers can choose both prices as well as capacity.

APPENDIX

Proof: (Theorem 1) From the first order optimization conditions for Π_i, we get:

$$\begin{aligned} x_{1i} + p_{i1}x_{1i,p_{i1}} &= 0 \\ x_{2i} + p_{i2}x_{2i,p_{i2}} &= 0 \\ p_{i1}x_{1i,\alpha_i} - p_{i2}x_{2i,\bar{\alpha}_i} &= 0 \end{aligned} \quad (18)$$

where and $\bar{\alpha}_i = 1 - \alpha_i$. Using Lemma 2, we now study Nash equilibrium which satisfy the Wardrop equilibrium conditions in equation (4) with equality. From equation (4) we have:

$$V_1' = p_{i1} + d_{i1}(x_{1i}, \alpha_i), \quad (19)$$
$$V_2' = p_{i2} + d_{i2}(x_{2i}, \bar{\alpha}_i) \quad (20)$$

By taking partial derivatives in equation (19) w.r.t. p_{i1} and α_i and in (20) w.r.t p_{i2}, we get

$$V_1'' \sum_j x_{1j,p_{i1}} = \begin{cases} 1 + d_{j1,x_{1j}} x_{1j,p_{i1}}, & j = i \\ d_{j1,x_{1j}} x_{1j,p_{i1}}, & j \neq i \end{cases} \quad (21)$$

$$V_2'' \sum_j x_{2j,p_{i2}} = \begin{cases} 1 + d_{j2,x_{2j}} x_{2j,p_{i2}}, & j = i \\ d_{j2,x_{2j}} x_{2j,p_{i2}}, & j \neq i \end{cases} \quad (22)$$

$$V_1'' \sum_j x_{1j,\alpha_i} = \begin{cases} d_{j1,x_{1j}} x_{1j,\alpha_i} + d_{j1,\alpha_i}, & j = i \\ d_{j1,x_{1j}} x_{1j,\alpha_i}, & j \neq i \end{cases} \quad (23)$$

$$V_2'' \sum_j x_{2j,\bar{\alpha}_i} = \begin{cases} d_{j2,x_{2j}} x_{2j,\bar{\alpha}_i} + d_{j2,\bar{\alpha}_i}, & j = i \\ d_{j2,x_{2j}} x_{2j,\bar{\alpha}_i}, & j \neq i \end{cases} \quad (24)$$

Define $X_{p_{ik}} = (x_{kj,p_{ik}}; j = 1, \ldots, n)', k = 1, 2$, $X_{\alpha_i} = (x_{1j,\alpha_i}; j = 1, \ldots, n)'$ and $X_{\bar{\alpha}_i} = (x_{2j,\bar{\alpha}_i}; j = 1, \ldots, n)'$. Then we have from equations (21)-(24) with b_{p_i} a vector with all zeros except 1 at the ith entry, b_{α_i} a vector with all zeros except d_{i1,α_i} at the ith entry and $b_{\bar{\alpha}_i}$ a vector with all zeros except $d_{i2,\bar{\alpha}_i}$ at the ith entry,

$$\begin{aligned} X_{p_{ik}} &= -(D_k - V_k''U)^{-1} b_{p_i}, \ k = 1, 2 \\ X_{\alpha_i} &= -(D_1 - V_1''U)^{-1} b_{\alpha_i} \\ X_{\bar{\alpha}_i} &= -(D_2 - V_2''U)^{-1} b_{\bar{\alpha}_i} \end{aligned} \quad (25)$$

To show the existence of $X_{p_{ik}}, X_{\alpha_i}$ and $X_{\bar{\alpha}_i}$ we need to show that $(D_k - V''U)^{-1}$ is invertible. We state a result on the inverse of the sum of matrices.

Lemma 5 ([13]): Let B be a nonsingular matrix and u and v' be column and row vectors respectively, then

$$\{B + buv'\} = B^{-1} - \frac{b}{1 + bv'B^{-1}u} B^{-1}uv'B^{-1}.$$

Using this Lemma, we can write:

$$(D_k - V_k''U)^{-1} = - \left[D_k^{-1} + \frac{V_k''}{1 - V_k'' \sum_{j=1}^N d_{jk,x_{kj}}^{-1}} D_k^{-1} I_1 D_k^{-1} \right].$$

Define $\Delta_k = 1 - V_k'' \sum_{j=1}^N d_{jk,x_{kj}}^{-1}$ and $\Delta_{k,-i} = 1 - V_k'' \sum_{j=1, j\neq i}^N d_{jk,x_{kj}}^{-1}$. With $d_{jk,x_{kj}} > 0$ and $V'' \leq 0$ (by assumption) we have $\Delta_k > 0$. Thus we need to establish the existence of D_k^{-1} which follows as D_k is a diagonal matrix with all positive entries (as $d_{jk,x_{kj}} > 0$), it is invertible. Since D_k^{-1}, $k = 1, 2$ exists we have existence of $X_{p_{ik}}, X_{\alpha_i}$ and $X_{\bar{\alpha}_i}$ from equations (25). Without loss of generality we take $i = 1$. Specifically, we have:

$$X_{p_{1k}} = - \begin{bmatrix} d_{1k,x_{k1}}^{-1} \frac{\Delta_{k,-1}}{\Delta_k} \\ d_{1k,x_{k1}}^{-1} \frac{d_{2k,x_{k2}}^{-1}}{\Delta_k} \\ \vdots \end{bmatrix} \quad (26)$$

$$X_{\alpha_1} = - \begin{bmatrix} d_{11,x_{11}}^{-1} d_{11,\alpha_1}^{-1} \frac{\Delta_{1,-1}}{\Delta_1} \\ d_{11,x_{11}}^{-1} d_{11,\alpha_1}^{-1} \frac{V'' d_{21,x_{12}}^{-1}}{\Delta_1} \\ \vdots \end{bmatrix} \quad (27)$$

$$X_{\bar{\alpha}_1} = - \begin{bmatrix} d_{12,x_{21}}^{-1} d_{12,\bar{\alpha}_1}^{-1} \frac{\Delta_{2,-1}}{\Delta_2} \\ d_{12,x_{21}}^{-1} d_{12,\bar{\alpha}_1}^{-1} \frac{V'' d_{22,x_{22}}^{-1}}{\Delta_2} \\ \vdots \end{bmatrix} \quad (28)$$

Writing the first order conditions from equations (18) in matrix form with $X_k = (x_{ki}, i = 1, \ldots, n)'$ and $P_k = (p_{ik}, i = 1 \ldots, n)'$, we have

$$X_k + \mathcal{X}_{p_{1k}} P_k = 0, \ k = 1, 2 \quad (29)$$
$$\mathcal{X}_{\alpha_1} P_1 - \mathcal{X}_{\bar{\alpha}_1} P_2 = 0, \quad (30)$$

where $\mathcal{X}_{p_{1k}} = diag\left(d_{1k,x_{k1}}^{-1} \frac{\Delta_{k,-1}}{\Delta_k}, d_{1k,x_{k1}}^{-1} \frac{d_{2k,x_{k2}}^{-1}}{\Delta_k}, \ldots \right)$ and $\mathcal{X}_{\alpha_1}$ is

$$diag\left(d_{11,x_{11}}^{-1} d_{11,\alpha_1}^{-1} \frac{\Delta_{1,-1}}{\Delta_1}, d_{11,x_{11}}^{-1} d_{11,\alpha_1}^{-1} \frac{V'' d_{21,x_{12}}^{-1}}{\Delta_1}, \ldots \right)$$

and $\mathcal{X}_{\bar{\alpha}_1}$ is

$$diag\left(d_{12,x_{21}}^{-1} d_{12,\bar{\alpha}_1}^{-1} \frac{\Delta_{2,-1}}{\Delta_2}, d_{12,x_{21}}^{-1} d_{12,\bar{\alpha}_1}^{-1} \frac{V'' d_{22,x_{22}}^{-1}}{\Delta_2}, \ldots \right).$$

From equations (19), (20) and (29) we get

$$X_k = -\mathcal{X}_{p_{1k}} (V_k' \mathbb{I} - d_k), \quad (31)$$

and from equations (19), (20) and (30) we have

$$0 = \mathcal{X}_{\alpha_1}(V_1' \mathbb{I} - d_1) - \mathcal{X}_{\bar{\alpha}_1}(V_2' \mathbb{I} - d_2), \quad (32)$$

where $d_k = (d_{ik}, i = 1, \ldots N)$. From equations (31) and (32) we have, with $Y = (\alpha_i, i = 1, \ldots, N)$,

$$X_1 = G_1(X_1, X_2, Y), \tag{33}$$
$$X_2 = G_2(X_1, X_2, Y), \tag{34}$$
$$0 = G_3(X_1, X_2, Y) \tag{35}$$

with $G_i(X_1, X_2, Y) : \mathbb{R}^{3n} \rightarrow \mathbb{R}$, for $i = 1, 2, 3$, $G_1(X_1, X_2, Y) := -\mathcal{X}_{p_{1k}}(V_k'\mathbb{I} - d_k)$, $G_2(X_1, X_2, Y) := -\mathcal{X}_{p_{1k}}(V_k'\mathbb{I} - d_k)$ and $G_3(X_1, X_2, Y) := \mathcal{X}_{\alpha_1}(V_1'\mathbb{I} - d_1) - \mathcal{X}_{\bar{\alpha}_1}(V_2'\mathbb{I} - d_2)$. First note that since V is concave, $\Delta_k > 0$ since the delay functions are strictly increasing. From the definition of $G_i(.)$, $\mathcal{X}_{p_1 k}$, $\mathcal{X}_{\alpha_1}, \mathcal{X}_{\bar{\alpha}_1}$, and the fact that d and V are continuously differentiable (by assumption), we can conclude that $G_i(.)$, $i = 1, 2$ are continuously differentiable.

Since G_3 is continuously differentiable, by implicit function theorem [19] (invertibility of the Jacobian of $G_3(X_1, X_2, Y)$ in general, can be established), there exists an $F_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t $\{(X_1, X_2, F_1(X_1, X_2)\} = \{(X_1, X_2, Y) : G_3(X_1, X_2, Y) = 0\}$ which implies we have

$$Y = F_1(X_1, X_2) \tag{36}$$

s.t. $(X_1, X_2, Y) \in U \times V \times W$ where U, V and W are open sets. Then, there exist closed and bounded sets $\bar{U}, \bar{V}$ and $\bar{W}$ s.t., $\bar{U} \subset U$, $\bar{V} \subset V$ and $\bar{W} \subset W$. Since G_1, G_2 and F_1 are continuous functions and $X_1, X_2, Y \in \bar{U}, \bar{V}, \bar{W}$ which are compact, then by Brouwer's fixed point theorem there exists an (X_1^*, X_2^*, Y^*) which satisfies equations (33), (34) and (36). Then, from equation (19) and (20) we get P^*. This establishes the existence of a fixed point (X_1^*, X_2^*, Y^*, P^*). ■

Proof: (Theorem 3) As proved in [23], [21], the price of anarchy for general concave utility and convex delay functions is lower bounded by price of anarchy for linear-truncted utility functions, and affine delay functions. We thus work with linear affine delay functions. Let S_t denote the social welfare in the game with linear truncated utility function, S_t^* the social welfare at Nash equilibrium and p_t^* be the price and α_t^* the processor sharing ratio vector at Nash equilibrium.

Lemma 6 ([21]): Under assumption on continuity and differentiability of V_k and d_{ik}, $\forall i$, we have

$$\min_{p_t^*, \alpha_t^*} \frac{S_t(p_t^*, \alpha_t^*)}{S_t^*} \geq \min_{p^*, \alpha^*} \frac{S(p^*, \alpha^*)}{S^*}.$$

For the PoA analysis, we work with convex delay functions. Let $\bar{V}_k$ be the Y-intercept of the tangent to the disutility curve at V_k'. Figure 2 characterizes the Nash

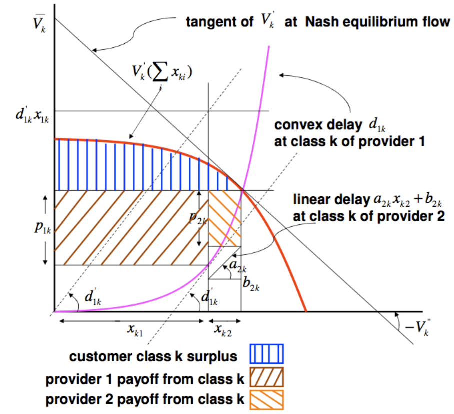

Fig. 2. Providers with both convex and linear delays. $\bar{V}_k$ is dependent on the processor sharing ratio α_i^* at Nash equilibrium.

equilibrium flow. At Nash equilibrium, the flows x_{ki} and prices p_{ik} of any provider i satisfy

$$\begin{aligned} \bar{V}_k + V_k'' \sum_i x_{ki} &= d_{ik} + p_{ik} \\ &\leq d_{ik}' x_{ki} + p_{ik} \end{aligned} \tag{37}$$

where $\bar{V}_k = -V_k'' \sum_i \alpha_{ik} y_i$, where $\alpha_{ik} y_i$ is the service rate of class k of provider i. If d_{ik}' is affine we have equality in (37) for d_{ik} of the form $a_{ik} x_{ki}$. From (37) and (13) we have

$$\bar{V}_k + V_k'' \sum_i x_{ki} = 2d_{ik}' x_{ki} + \delta_{ik} x_{ki} \tag{38}$$

Writing (38) in matrix form:

$$\bar{V}_k \leq (-V_k'' M + 2B_k + \Delta_k) X_k, \tag{39}$$

where, B_k and Δ_k are order N diagonal matrices with

$$B_k = \begin{bmatrix} d_{1k}' & 0 & \cdots & 0 \\ 0 & d_{2k}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{Nk}' \end{bmatrix},$$

$$\Delta_k = \begin{bmatrix} \delta_{1k} & 0 & \cdots & 0 \\ 0 & \delta_{2k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{Nk} \end{bmatrix},$$

M is a order N square matrix of all ones. Similarly for social optimum flow X_k^* we have

$$\bar{V}_k \leq (2B_k^* - V_k'' M) X_k^*, \tag{40}$$

where B_k^* is order N diagonal matrix of $d_{ik}'^*$ which are delay derivatives evaluated at social optimum. For linear disutility (not truncated) the social welfare at Nash, S, is given by

$$S = \sum_k X_k^T \left(B_k - \frac{1}{2} V_k'' M + \Delta_k \right) X_k \quad (41)$$

For linear truncated disutility the social welfare at Nash, S_t is given by

$$S_t = \sum_k X_k^T (B_k + \Delta_k) X_k \quad (42)$$

Let S_l^* be the social welfare at the social optimum configuration for a linear disutility.

$$\begin{aligned} S_t^* &= S_l^* + \sum_k \frac{V_k''}{2} X_k^T M X_k \quad (43) \\ &= \sum_k {X_k^*}^T (B_k - \frac{1}{2} V_k'' M) X_k^* + \frac{V_k''}{2} X_k^T M X_k \end{aligned}$$

From (42) and (43) we can write,

$$\begin{aligned} & 3S_t - 2S_t^* = \quad (44) \\ & \quad \sum_k X_k^T (3B_k + 3\Delta_k - V_k'' M) X_k \\ & \quad -(\bar{V}_k^T + \mathcal{D}_k^T)(2B_k^* - V_k'' M)^{-1} (\bar{V}_k + \mathcal{D}_k) \\ & = \sum_k X_k^T (2B_k^* - B_k + \Delta_k) X_k \\ & \quad -X_k^T (2(B_k - B_k^*) + \Delta_k)(2B_k^* - V_k'' M)^{-1} \\ & \quad (2(B_k - B_k^*) + \Delta_k) X_k \\ & \quad -X_k^T (-V_k'' M + 2B_k + \Delta_k) \mathcal{E}_k \\ & \quad -\mathcal{E}_k^T (X_k + (2B_k^* - V_k'' M)^{-1} \\ & \quad (2(B_k - B_k^*) + \Delta_k) X_k + \mathcal{E}_k) \end{aligned}$$

For affine delays $\mathcal{C}_k = \mathcal{D}_k = \mathcal{E}_k = 0$ and $B_k^* = B_k$. Thus (44) reduces to

$$\begin{aligned} & 3S_t - 2S_t^* = \\ & \sum_k X_k^T \left[B_k + \Delta_k - \Delta_k (2B_k - V_k' M)^{-1} \Delta_k \right] X_k. \end{aligned}$$

Since $X_k \geq 0$, we need to show that all entries in the matrix insided the square bracket in (44) are positive so that LHS is positive. This can be established in the same way as in [21]. ∎

References

[1] E. Altman, D. Barman, R. El-Azouzi, D. Ros, and B Tuffin, "Pricing Differentiated Services: a game theoretic approach", *Proc.of IFIP Networking*, p.430441, May 2004.

[2] D. Acemoglu and A. Ozdagler, "Competition and efficiency in congested markets", *Mathematics of Operations Research*, 32(1):1-31, 2007.

[3] C. Courcoubetis and R. Weber, *Pricing Communication Networks: Economics, Technology and Modelling*, Wiley Interscience, 2003.

[4] P. Dube, C. Touati and L. Wynter, "Capacity Planning, Quality of Service and Price Wars", *ACM Sigmetrics Performance Evaluation Review*, 35:3, 31-33, 2007.

[5] P. Dube and R. Jain, "N-player Bertrand and Cournot queueing games: Existence of equilibrium", *Proc. Allerton Conference*, 2008.

[6] N. Edelson and D. Hildebrand, "Congestion tolls for Poission queueing processes", *Econometrica*, 43(1):81-92, 1975.

[7] J.M. Harrison, "Dynamic scheduling of a multi-class queue: Discount optimality", *Operations Research* 23(2):270-282, 1975.

[8] R. Hassin and M. Haviv, *To Queue or Not to Queue*, Kluwer Academic Publishers, 2003.

[9] A. Hayrapteyan, E. Tardos and T. Wexler, "A network pricing game for selfish traffic", *ACM SIGACTS-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, 2005.

[10] M. Haviv, "The Aumann-Shapley price mechanism for allocating congestion costs", *Operations Research Letters* 29:211-215, 2001.

[11] R. Jain and P. Dube, "Queueing game models for differentiated services", *Proc. Int. Conf. on Game Theory in Networks (GameNets)*, 2009.

[12] R. Johari, G. Weintraub and B. Van Roy, "Investment and Market Structure in Industries with Congestion", *Operations Research*, 2009.

[13] K. S. Miller, "On the Inverse of the Sum of Matrices", *Mathematics Magazine*, Vol. 54, No. 2 (Mar. 1981), pp. 67-72.

[14] C. Loch, "Pricing in markets senstitive to delay", *Ph.D. Dissertation*, Stanford University, 1991.

[15] I. Luski, "On partial equilibrium in a queuing system with two servers", *Review of Economic Studies* 43(3):519-525, 1976.

[16] J. Mackie-Mason and H. Varian, "Pricing the Internet", in *Public Access to the Internet*, B. Kahin and J. Keller, (eds.), pp. 269-314, 1995, MIT Press.

[17] C. Maglaras and A. Zeevi, "Pricing and design of differentiated services: Approximate analysis and structural insights", *Operations Research*, 53(2):242-262, 2005.

[18] P. Marbach, "Pricing Differentiated Services Networks: Bursty Traffic", *Proc. INFOCOM*, 2001.

[19] A. Mas-Colell, M. Whinston and J. Green, *Microeconomic Theory*, Oxford University Press, 1995.

[20] H. Mendelson and S. Whang, "Optimal incentive-compatible priority pricing for the M/M/1 queue", *Operations Research*, 38(5):870-883, 1990.

[21] J. Musacchio and S. Wu, "The price of anarchy in a network pricing game", *Proc. of 45th Annual Allerton Conference*, September 2007.

[22] P. Naor, "The regulation of queue size by levying tolls", *Econometrica*, 37(1):15-24, 1969.

[23] A. Ozdagler, "Price competition with elastic traffic", *Networks*, 2006.

[24] A. Odlyzko, "Paris Metro pricing for the Internet", *Proceedings of ACM conference on electronic commerce*, pp. 140147, 1999.

[25] A. Orda and N. Shimkin, "Incentive Pricing in Multi-Class Communication Networks", *Proc. IEEE INFOCOM*, 1997.

[26] J. Shu and P. Varaiya, "Pricing Network Services", *Proc. IEEE INFOCOM*, 2003.

[27] D. Stahl and A. Whinston, "A general economic equilibrium model of distributed computing", in *New directions in computational economics*, eds. W. Cooper and A. Whinston, Kluwer Acad. Pub., pp.175-189, 1994.